Regenerative Receivers Using Capacitive Feedback

Dipl.-Phys. Jochen Bauer

02/08/2014

Abstract

The design concept of a regenerative receiver using a capacitive feedback path back into the grid tank is well known because is was used in the first regenerative receiver design patented by E.H. Armstrong in 1913. As opposed the the more prevalent inductive feedback employing a "tickler coil", capacitive feedback in most cases relies on the internal plate to grid capacitance of vacuum triodes and requires no external components to complete the feedback path. Most intriguing with capacitive feedback is the issue of exercising feedback control. Since the internal plate to grid capacitance of the triode in use cannot be altered, feedback control needs to be implemented by an adjustable tuned plate tank in the regenerative RF amplifier stage. The two basic plate circuits here are the parallel LC plate tank and the series LC plate tank and we will analyze the behavior of regenerative RF amplifier stages based on capacitive feedback for both of these by deriving equations valid for an arbitrary LC network in the plate circuit. As is turns out, the parallel LC plate tank typically offers an abundance of positive feedback but suffers from a highly sensitive feedback control and has a very pronounced detuning effect on the grid tank. The series LC tank, in contrast, offers a very fine-grained feedback control with less detuning but there may be circumstances where it might not have enough positive feedback reserves.

Introduction And History

The regenerative receiver has been the prevalent circuit for radio receivers for many years in the early history of wireless communications. Even today, although having been superseded by the superheterodyne receiver many decades ago it is still appreciated by radio amateurs for it's simplicity and often surprising performance, in particular in the shortwave frequency range. This type of receiver uses positive feedback from the output of an RF amplification stage to the input (grid) tank[1] of this stage. This positive feedback can increase the virtual [2] Q-factor of the grid tank well over it's physical value and greatly help with

[1]In vacuum tube circuits the input tank is often referred to as the grid tank

[2]The virtual Q-factor is the apparent Q-factor of the tank under the influence of feedback

the reception of weak radio signals [1],[2].

Although most regenerative circuits use inductive feedback by a tickler coil [1], [2], capacitive feedback through the internal plate to grid capacitance of a vacuum tube triode has been known since the very early days of radio history. In fact, the first regenerative receiver invented by E.H. Armstrong [3] in 1913 relied on capacitive feedback through the internal plate to grid capacitance of the vacuum tube triode he used. Although intended as an oscillator rather than a regenerative receiver, L. Kühn filed a patent application in 1917 for a circuit using the same feedback mechanism [4]. Let us therefore take a closer look at this type of feedback while paying particular attention to it's application in regenerative circuits.

Regenerative receivers of both types (inductive feedback and capacitive feedback) have an important issue in common: Feedback control. First, we need to control the sign of the feedback: Positive feedback will increase the virtual Q-factor of the grid tank and is desired while negative feedback will lower the virtual Q-factor of the grid tank and is undesired. Second, we need to control the amount of (positive) feedback. Not enough positive feedback will leave the grid tank at a low virtual Q-factor while too much positive feedback will cause the receiver to oscillate. In case of inductive feedback by a tickler coil, feedback control is straight forward. The relative orientation of the turns of the tickler coil to the turns of the grid tank coil can be used to determine the sign of the feedback and the most simple method to control the amount of positive feedback is varying the distance between the tickler coil and the grid tank coil. With capacitive feedback, exercising this kind of control turns out to be a little more complicated. This is because unlike the tickler coil we have no control over the internal plate to grid capacitance of the vacuum tube triode being used. It was found experimentally by Armstrong that a tuned plate circuit could be used for feedback control by varying it's resonant frequency. As it turns out, this tuned circuit can either be a series LC tank or a parallel LC tank. The basic circuit of both versions is depicted in figure 1.

It should be noted that since the positive DC supply voltage is typically AC-grounded, the series LC plate tank version of the circuit needs an RF choke in the DC supply path to avoid shunting the variable capacitor to ground. If, however, the DC supply is a battery with a relatively high RF impedance then the RF choke can be omitted and the positive terminal of the battery is connected directly to the point in the circuit where the inductor and the variable capacitor meet, while it's negative terminal is grounded. In fact, early sketches by Armstrong of his regenerative receiver showed such a setup [5] [3].

[3]This setup is, of course, prone to being misinterpreted as a variable capacitor being used to create an RF-grounding for the positive supply voltage. However, the capacitor is actually a part of the series LC plate tank

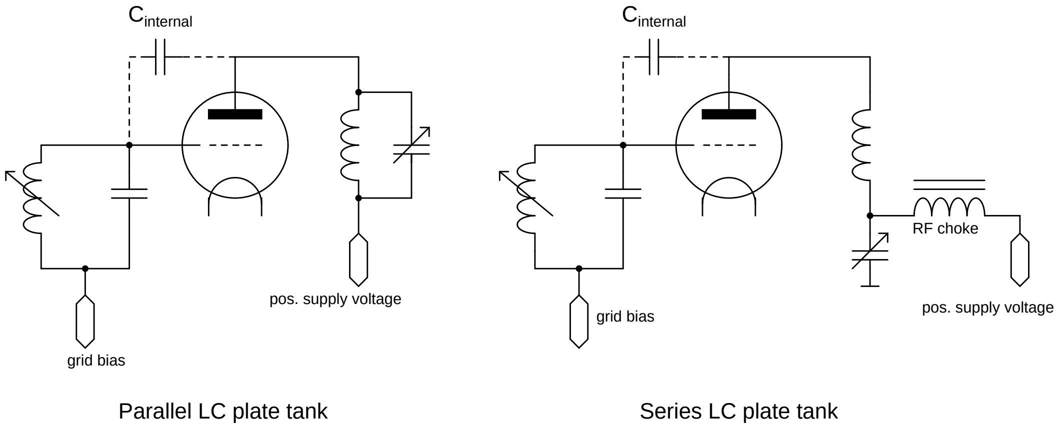

Figure 1: Capacitive feedback using internal tube capacitances

As already mentioned, capacitive feedback control behaves in no way as straight forward as feedback control with tickler coils. Simply trying to put a (variable) inductor into the plate circuit just the same way as one would connect the tickler coil can lead to quite disappointing results as demonstrated in [6]. However, when looking at the theoretical background of capacitive feedback, the behavior of this type of regenerative receiver becomes a lot clearer and we shall do so in the following section.

Analysis Using The Small Signal Triode Model

To get a basic understanding of how capacitive feedback works and how it can be controlled, we will look at a tuned grid RF amplifier using a vacuum tube triode. Using the triode's linear small signal model and also modeling the losses in the grid tank coil by a series loss resistor [7] we immediately arrive at the equivalent circuit given in figure 2.

The grid tank is comprised of an inductor L and a capacitor C. The coil losses are modeled by a loss resistor R connected in series with the coil [7]. The antenna voltage coupled inductively[4] into the grid tank is represented by a sinusoidal driving voltage U_0 at (angular) frequency ω. The grid resistor R_g is assumed to be sufficiently large so that the current it draws can be neglected ($I_{Rg} = 0$) [5] The input voltage U_g at the grid of the triode is equal to the voltage U_C developing across the capacitor C. In the linear small signal model, the triode is replaced by

[4]This implies that the voltage source is "in" the coil, hence the voltage source in this model needs to be connected directly to the coil

[5]Strictly speaking, this implies the assumption $\omega \gg \frac{1}{R_g C}$. However, for usual values of R_g and C this is of no concern for RF input frequencies

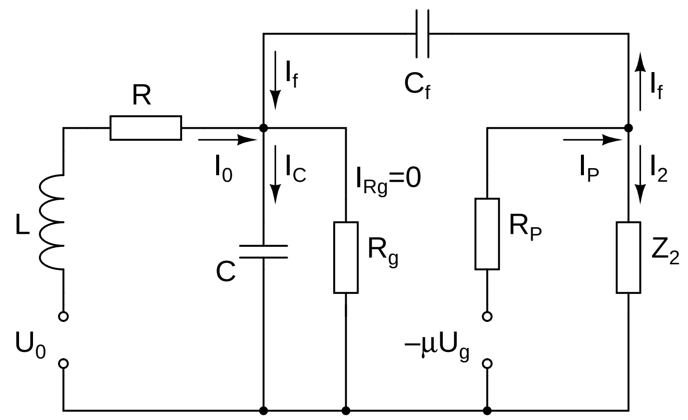

Figure 2: Equivalent circuit using the triode small signal model

a voltage source with a (real-valued) output impedance R_P and an open circuit voltage of $-\mu U_g$. The negative sign accounts for the inversion between input and output voltage while the open circuit voltage amplification factor μ is always positive. Note that R_P is the *internal* plate resistance of the triode. The internal plate to grid capacitance of the triode is described by the feedback capacitor C_f.

Connected to the plate is a complex plate load Z_2. In [8] an analysis is given for the special case of the complex plate load being a parallel LC tank. Here, we will aim at deriving equations that are valid for an arbitrary network of inductors and capacitors in the plate circuit and we will also look at grid tank detuning due to feedback. Furthermore, by taking the tuned grid circuit into account, we obtain slightly more precise expressions for the conditions under which positive feedback occurs.

The total complex impedance Z that the voltage source U_0 in the grid tank sees is given by

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) + Z_f \tag{1}$$

where Z_f is the additional complex impedance that appears in the grid tank due to feedback. We assume that the losses in the LC network in the plate circuit are neglectable, i.e. Z_2 is purely imaginary and can therefore be written as

$$Z_2 = \frac{j}{B} \tag{2}$$

The real part of the additional complex impedance $\mathrm{Re}Z_f$ which is the additional (ohmic) resistance R_f showing up in the grid tank is then given by

$$R_f = \mathrm{Re}Z_f = \frac{\mu R_P \left(1 - \frac{1}{\omega C_f} B\right)}{\left(\mu + \frac{C}{C_f}\right)^2 + (\omega C R_P)^2 \left(1 - \frac{1}{\omega C_f} B\right)^2} \tag{3}$$

while the imaginary part of the additional complex impedance $\mathrm{Im}Z_f$ which is the additional reactance X_f showing up in the grid tank is given by

$$X_f = \mathrm{Im}Z_f = \frac{\frac{\mu}{\omega C}\left(\mu + \frac{C}{C_f}\right) - R_P^2 B \left(1 - \frac{1}{\omega C_f} B\right)}{\left(\mu + \frac{C}{C_f}\right)^2 + (\omega C R_P)^2 \left(1 - \frac{1}{\omega C_f} B\right)^2} \tag{4}$$

The reader is referred to **Appendix A** for a detailed derivation of the above equations [6]. Note that in general, B will depend on the driving frequency ω. The additional reactance X_f in the grid tank causes an undesired detuning of the grid tank, while the additional resistance R_f, depending on it's sign, can either decrease or increase the virtual Q-factor of the grid tank. We shall take a closer look at this influence on the virtual Q-factor in the following section

Loss Resistance And Q-factor Of The Grid Tank

In the previous section, we have used a series loss resistor R to account for the losses occurring in the grid tank coil L. Since in most practical tuned circuits the capacitor losses can be neglected against the coil losses, the Q-factor of the grid tank without feedback[7] is [7]

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

From equations (1) and (3) it follows that the influence of feedback can be taken into account by replacing R with $R + R_f$. Hence, the apparent Q-factor of the grid tank with feedback is

$$Q' = \frac{1}{R + R_f}\sqrt{\frac{L}{C}}$$

Throughout this paper, we have dubbed Q' the *virtual* Q-factor of a tuned circuit. It can immediately be seen that our goal of increasing the virtual Q-factor of the grid tank above it's physical value requires the additional resistance R_f to be negative. Based on the sign and phase relations between driving and feedback voltage in the grid tank, the type of feedback that leads to a negative additional

[6] R_f and X_f depend on the grid tank's capacitance C but not on it's inductance L, this is a consequence of the voltage source being "in" the coil

[7] This a also referred to as the *physical* Q-factor

resistance R_f is referred to as *positive* feedback. In summary, we need positive feedback to create a negative additional resistance R_f in order to increase the virtual Q-factor Q' of the grid tank above it's physical value Q.

By looking at equation (3) we see that since $\mu > 0$ the additional resistance R_f will be negative if

$$\left(1 - \frac{1}{\omega C_f} B\right) < 0 \tag{5}$$

There is, of course, a natural limit for the amount of positive feedback that should reasonably be applied. If the amount of positive feedback is increased so that $R + R_f < 0$, the circuit will start to oscillate and can no longer be used as a (regular) regenerative receiver[8]. The condition for reasonable feedback is therefore obviously given by R_f being in the range from 0 to $-R$. From the above equations it follows that for tuned circuits with a physical Q-factor of typically between 40 and 100 as well as L/C ratios commonly used in the AM broadcast bands, the reasonable feedback range is roughly given by $0 > R_f > -50\Omega$.

We are now ready to explore the behavior of our model circuit for a parallel LC tank as well as a series LC tank in the plate circuit as depicted in figure 1. We shall start with the parallel LC tank.

Parallel LC Plate Circuit

For the complex impedance Z_2 of an inductance L_2 and a capacitance C_2 in a parallel setup, we have

$$\frac{1}{Z_2} = \frac{1}{j\omega L_2} + \frac{1}{\frac{1}{j\omega C_2}}$$

and hence

$$Z_2 = \frac{j}{\frac{1}{\omega L_2} - \omega C_2}$$

By looking at equation (2) we can immediately identify B as

$$B = \frac{1}{\omega L_2} - \omega C_2 \tag{6}$$

and from (5) the condition for positive feedback (R_f being negative) follows to be

[8]Provided that the amplitude of the oscillations is relatively small, the receiver might be able to perform synchronous detection

$$\frac{1}{L_2\,(C_2 + C_f)} > \omega^2 \qquad (7)$$

Since the feedback capacitance C_f, which is the internal plate to grid capacitance of the triode, will usually be small compared to the plate tank capacitance, we approximately have[9].

$$\frac{1}{L_2 C_2} > \omega^2$$

Hence, we obtain positive feedback, increasing the virtual Q-factor of the grid tank, for input frequencies ω that are below the resonant frequency of the parallel LC plate tank.

In order to visualize the frequency dependence of the feedback provided into the grid tank, we can plug B from equation (6) into equation (3) for R_f and plot R_f as a function of ω. This has been done in figure 3 with the following parameters: $R_p = 10\text{k}\Omega$, $\mu = 15$, $C_f = 5\text{pF}$, $C = 200\text{pF}$, $L_2 = 300\mu\text{H}$ and $C_2 = 100\text{pF}$.

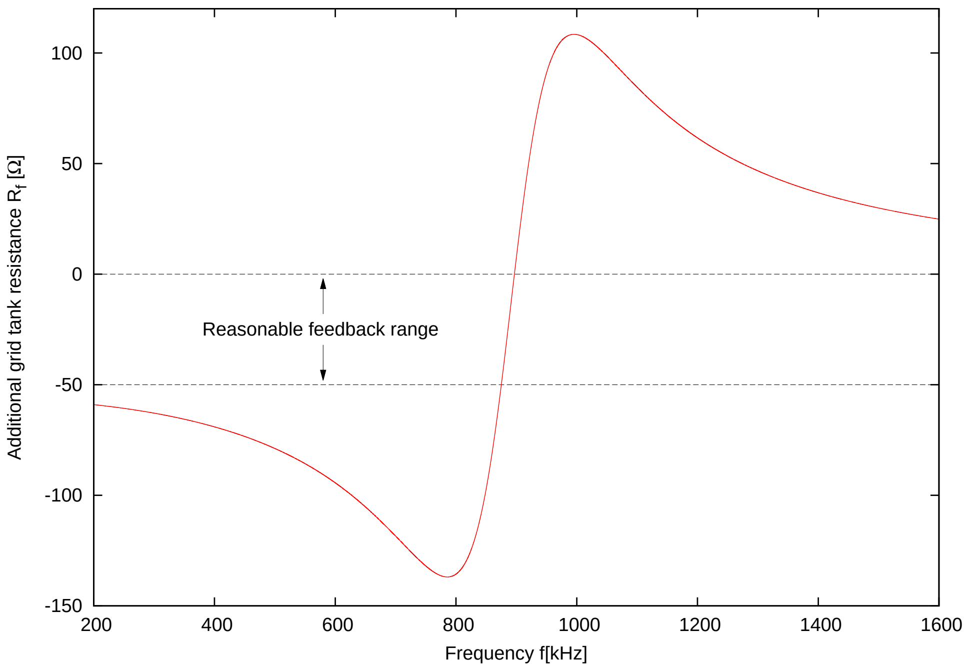

Figure 3: Additional grid tank resistance with parallel LC plate tank

For the convenience of the reader, the cycle per second frequency f in kHz has been used instead of the angular frequency ω. As expected, the additional resistance R_f is negative for input frequencies below the resonant frequency of the

[9] This result is also obtained in [8]

parallel LC plate tank (ca. 920kHz)[10]. Also, we can see that for a fixed setting of the plate tank, reasonable feedback as defined previously will only occur within a very small frequency range a little below the plate tank resonance.

Let us now turn to the task of feedback control. As has become obvious by now, the plate tank needs to be built with a variable inductor and/or variable capacitor because it's resonant frequency needs to be adjustable over the entire input frequency range. Otherwise, there would only be a very limited range of input frequencies for which reasonable feedback into the grid tank can be achieved.

Let us make the plate tank capacitor C_2 variable while keeping all other parameters as they are. Furthermore, in this example, we assume that we want to listen to a radio station broadcasting at a frequency of 900kHz. To achieve reasonable feedback into the grid tank for this input frequency, we need to adjust the resonant frequency of the plate tank to be just a little bit above 900kHz using the variable capacitor of the plate tank. To visualize this task, curves of the frequency dependence of the additional resistance R_f have been plotted for three different values of C_2. See figure 4.

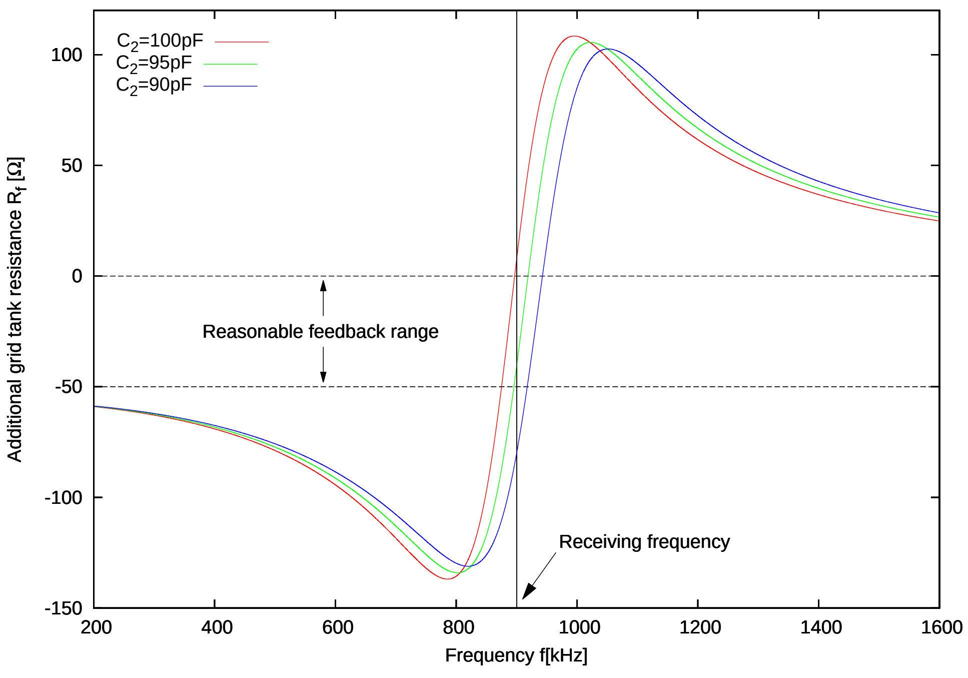

Figure 4: Adjusting feedback with a parallel LC plate tank

[10]This curve for the additional series resistance is consistent with the curve for the additional parallel input admittance given in [8]

It can be seen that for $C_2 = 100\text{pF}$ (red curve) we are above the reasonable feedback range for an input frequency of 900kHz. In fact, R_f is even slightly positive at the desired input frequency and the grid tank experiences adverse additional damping. For $C_2 = 95\text{pF}$ (green curve) we are within the reasonable feedback range for the desired input frequency with R_f being somewhere near -40Ω. For $C_2 = 90\text{pF}$ we are already clearly below the reasonable feedback range for our input frequency of 900kHz, thus providing too much positive feedback and the receiver might already have burst into oscillations.

Looking back, we realize that we have gone from slightly negative feedback to already too much positive feedback within a 10pF range of C_2. That is, the feedback control is highly sensitive and requires a steady hand for adjustment. This is clearly an adverse kind of behavior that originates in the steep slope of the curves near $R_f = 0$. It might therefore be tempting to try to use the region near the left margin of the plotting range instead of the region near $R_f = 0$ [11]. However, although this region would provide a much less sensitive feedback control, the feedback there is significantly below the reasonable feedback range [12] and will probably cause the receiver to oscillate.

Series LC Plate Circuit

In this case, the complex impedance Z_2 is given by

$$Z_2 = j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$$

and from equation (2) we obtain

$$B = \frac{1}{\omega L_2 - \frac{1}{\omega C_2}} \tag{8}$$

Inserting this expression into the condition for positive feedback given by equation (5) we obtain

$$\frac{1}{L_2 C_2} < \omega^2 < \frac{1}{L_2 \frac{C_2 C_f}{C_2 + C_f}} \tag{9}$$

At first, this condition seems more complicated than the corresponding expression in the case of a parallel LC plate tank. However, by looking at the top of the ω range, we see that this is simply the resonant frequency of L_2 in series with C_2 in series with C_f. Since the feedback capacitance C_f (internal plate to grid capacitance) is typically much lower than the capacitance C_2 that is chosen for the series LC plate tank, this simplifies to

[11] This has been suggested in [8]

[12] Even for frequencies as low as 10kHz, R_f will stay below -50Ω

$$\frac{1}{L_2C_2} < \omega^2 < \frac{1}{L_2C_f}$$

Hence, for reasonable values of L_2, the top of the ω range is of no concern here since the feedback capacitance C_f is usually very small.

If we now look at the bottom of the ω range, we see that this is exactly the opposite as for the parallel LC plate tank: The series LC plate tank needs to be operated below the input frequency ω to obtain positive feedback.

By using expression (8) for B in equation (3), we again obtain a function $R_f(\omega)$ giving the frequency dependence of the feedback provided into the grid tank. This function has been plotted in figure 5 with the same parameters as used in the previous section expect C_2 which has here been set to $C_2 = 150\text{pF}$.

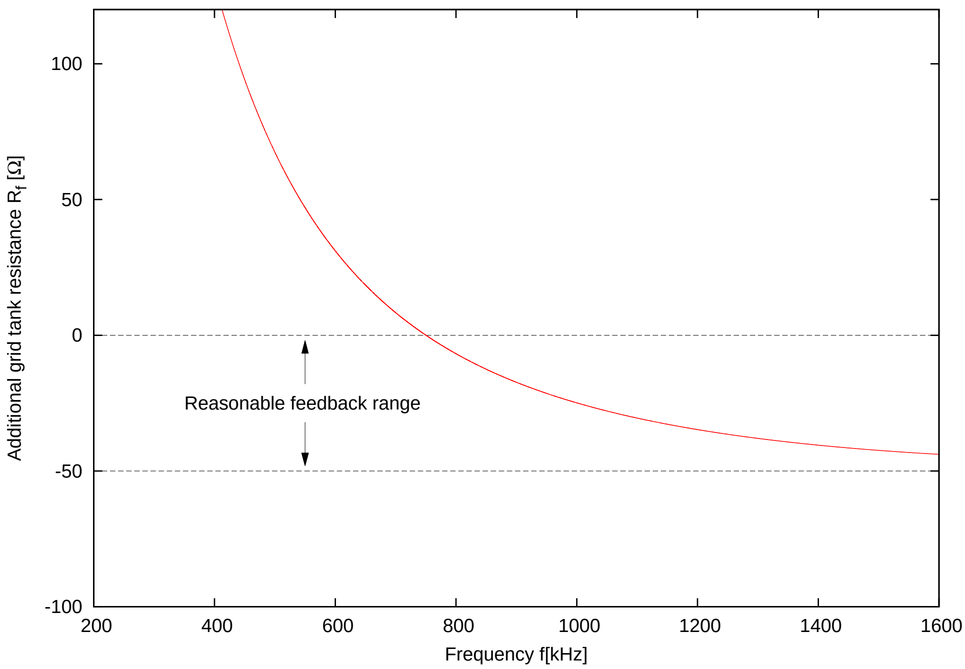

Figure 5: Additional grid tank resistance with series LC plate tank

Again, for the convenience of the reader the cycle per second frequency f in kHz has been used instead of the angular frequency ω. As expected, the additional resistance R_f is now negative for input frequencies above the resonant frequency of the series LC plate tank (ca. 750kHz). As opposed to the parallel LC plate tank we here have a quite large frequency range where there is reasonable feedback as defined above provided into the grid tank[13]. Obviously, this will make feedback

[13]In accordance with (9), R_f turns positive again at approximately 4.2Mhz

control much less sensitive, which is a desired effect.

Let us again make the plate tank capacitor C_2 variable while keeping all other parameters as they are and also let us assume again that we want to listen to a radio station broadcasting at 900kHz. Similar to the previous section, the use of the variable capacitor C_2 for the purpose of feedback control is visualized in figure 6.

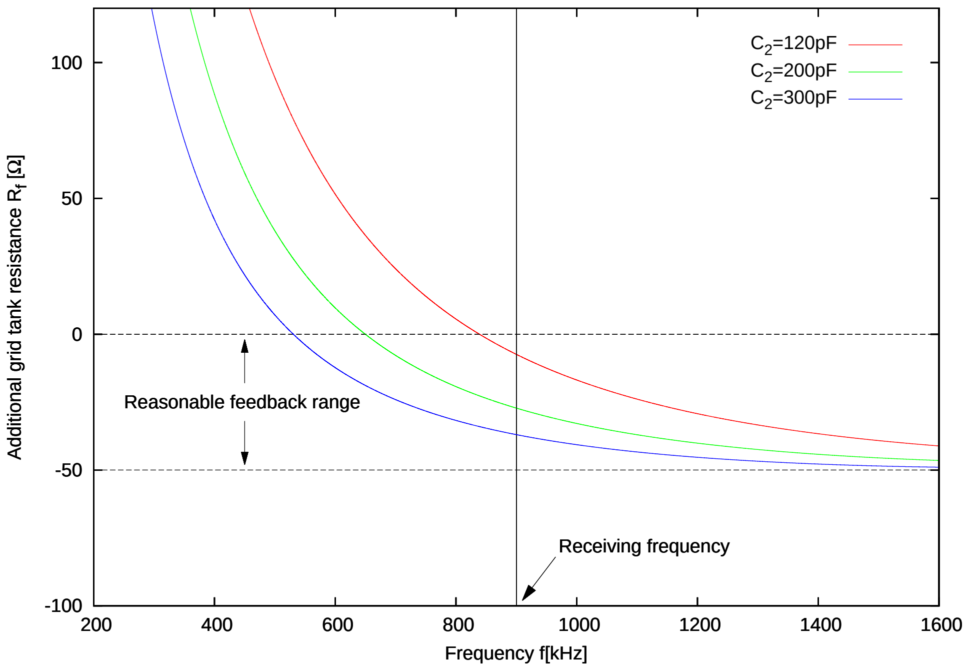

Figure 6: Adjusting feedback with a series LC plate tank

As expected, feedback control using C_2 is much less sensitive than in the case of a parallel LC plate tank and is therefore a lot easier to do in practice. In fact, we can vary C_2 between 120pF and 300pF and still stay in the reasonable feedback range. There is one problem though: To provide reasonable feedback in the entire range of possible input frequencies C_2, respectively it's variation range would have to be unreasonable large. However, this problem can simply be fixed by replacing L_2 with a variable inductor, greatly expanding the resonant frequency range of the plate tank. In fact, L_2 does not even have to be a continuously variable inductor, a tapped coil providing discrete inductances will do nicely.

While it seems to appear that the series LC plate tank is advantageous for feedback control, one problem remains. As it can be seen, the minimum of R_f for a series LC plate tank is not nearly as far in the negative region as for a parallel LC plate tank. Therefore, there might be situations where the receiver will not have enough positive feedback reserves for optimal performance.

Let us now turn our attention to the detuning of the grid tank caused by feedback which is an undesired side effect that requires constant re-adjustment of the grid tank's resonant frequency while operating the feedback control.

Detuning Of The Grid Tank

The additional complex impedance Z_f showing up in the main tank due to the application of feedback not only has a real part R_f but also an imaginary part X_f. Hence, the feedback that is applied will in general have a detuning effect on the grid tank. Taking this additional reactance X_f into account, the resonance condition of the grid tank becomes

$$\mathrm{Im}Z = \omega L - \frac{1}{\omega C} + X_f = 0$$

which has a solution ω that deviates from the resonant frequency without feedback given by $\omega_0 = \sqrt{1/LC}$. Let us now take a closer look at this detuning effect for a parallel and a series LC plate tank. As a measure of how pronounced the detuning effect on the grid tank is, we are looking at the "initial" detuning at frequencies where $R_f = 0$.

Parallel LC Plate Tank

In case of a parallel LC plate tank, there is only one zero of $R_f(\omega)$. It occurs at

$$1 - \frac{1}{\omega C_f} B(\omega) = 0$$

with $B(\omega)$ given by equation (6). This condition amounts to

$$\omega^2 = \frac{1}{L_2 \left(C_2 + C_f\right)}$$

which is consistent with (7) [14]. Substituting this result into equation (4) then yields

$$X_f = \frac{1}{\omega C} \cdot \frac{1}{1 + \frac{C}{\mu C_f}}$$

The resonance condition $\mathrm{Im}Z = 0$ then gives

$$\omega L - \frac{1}{\omega C} + \frac{1}{\omega C} \cdot \frac{1}{1 + \frac{C}{\mu C_f}} = 0$$

which is solved by

[14] Note that we need to use the more rigorous version of the positive feedback condition here

$$\omega = \sqrt{\frac{1}{LC}} \cdot \sqrt{1 - \frac{1}{1 + \frac{C}{\mu C_f}}}$$

From the above equation, we see that there is a non-zero initial detuning of the grid tank that occurs even when $R_f = 0$. Using the parameters $C = 200\text{pF}$, $C_f = 5\text{pF}$ and $\mu = 15$ from the previous sections, we obtain a relative initial detuning of 15% towards lower frequencies.

This can be understood by the fact that for $R_f = 0$, the parallel plate tank needs to be operated near it resonant frequency. Hence it provides a high load impedance Z_2 at the output of the triode stage, reducing it's output voltage that is going into the feedback path very little. However, the phase of the feedback voltage is such that although $R_f = 0$, the additional reactance X_f provided into the grid tank is considerable. In fact, by plotting the additional resistance R_f and the additional reactance X_f into the same diagram (see figure 7), we can see that in the reasonable feedback range $R_f < 0 < -50\Omega$, the additional reactance X_f (and hence the detuning effect) is near it's maximum.

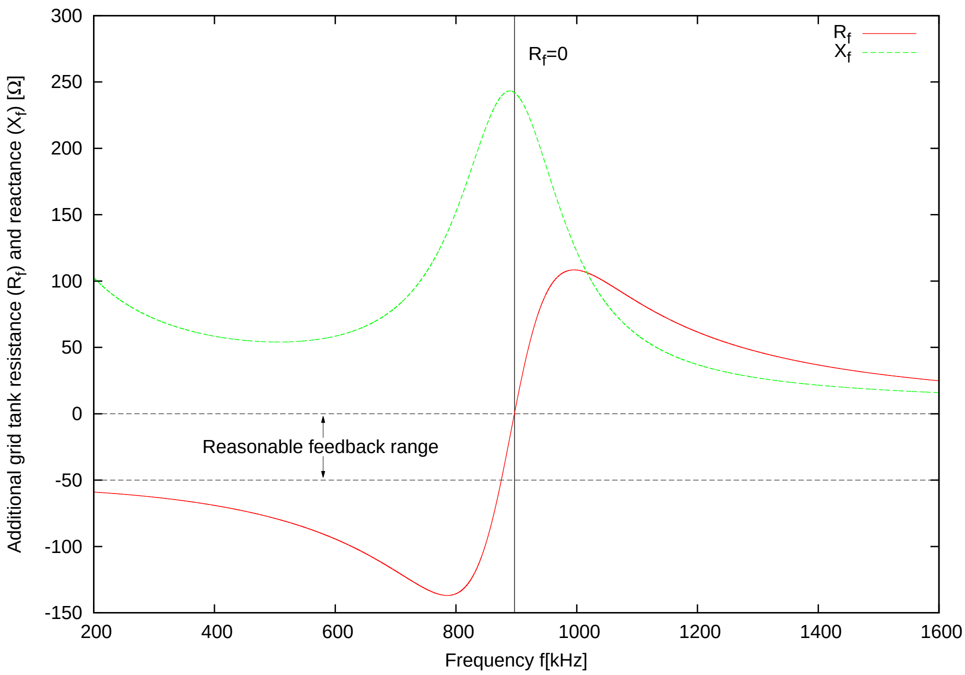

Figure 7: Additional grid tank resistance and reactance (parallel LC plate tank)

Series LC Plate Tank

In case of a series LC plate tank, there are two zeros of $R_f(\omega)$. The one that can be seen in figure 5 occurs because $R_f = 0$ for $B(\omega) \to \infty$. With $B(\omega)$ given by equation (8), this amounts to

$$\omega^2 = \frac{1}{L_2 C_2}$$

which is consistent with (9). Inserting this expression into (4) then yields

$$X_f = \frac{1}{\omega C} \cdot \frac{C_f}{C}$$

which since $C \gg C_f$ is a very small reactance[15] and the resonance condition $\mathrm{Im}Z = 0$ then turns into

$$\omega L - \frac{1}{\omega C} + \frac{1}{\omega C} \cdot \frac{C_f}{C} = 0$$

and since $C_f \ll C$ this becomes

$$\omega L - \frac{1}{\omega C} = 0$$

which, of course, leads to the unperturbed resonant frequency

$$\omega = \sqrt{\frac{1}{LC}}$$

of the grid tank. We see that the series LC plate tank does not cause any initial detuning of the grid tank. This result is pretty obvious since for $\omega L_2 - 1/\omega C_2$ the impedance Z_2 becomes zero and short-circuits the output of the triode stage. Hence, it's output voltage is zero and thus there is no feedback at all, leaving R_f and also X_f at zero. Again, we are interested in the amount of detuning that can be expected in the reasonable feedback range $R_f < 0 < -50\Omega$. Plotting R_f and X_f in the same diagram (see figure 8) as we did in the previous section reveals that in case of a series LC plate tank the reasonable feedback range conveniently falls onto a quite "flat" region near zero of the additional grid reactance X_f. Hence, there is very little detuning to be expected[16].

[15]As discussed later in this section, X_f should be exactly zero for a lossless series LC plate tank. The very small, but still non-zero value obtained here is due to the approximations made in the derivation of the expression for Z_f

[16]Note again that X_f is small but still non-zero at R_f which is, as mentioned earlier, due to the approximations made when deriving Z_f

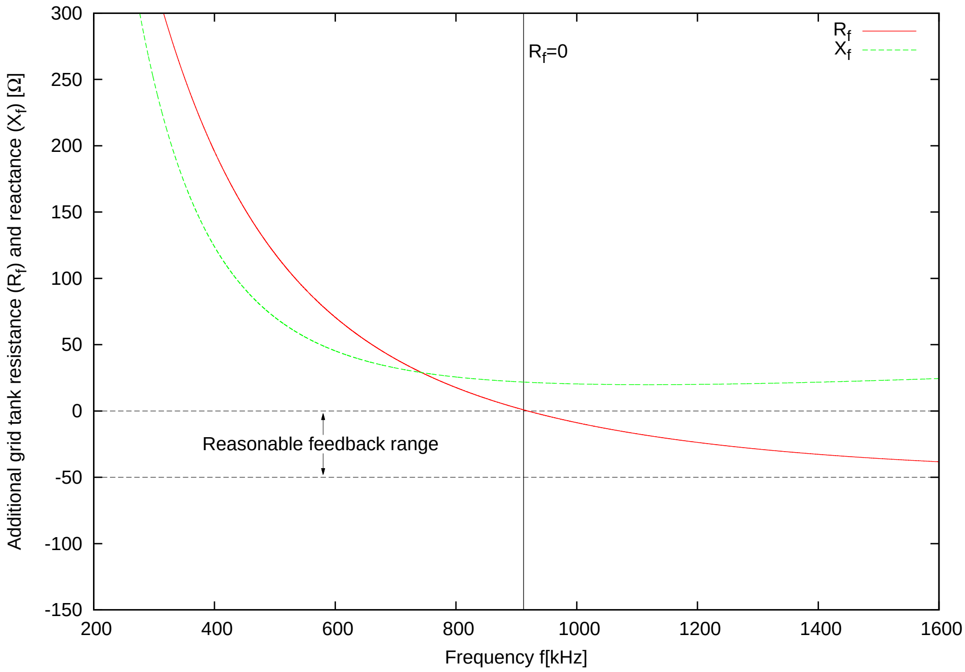

Figure 8: Additional grid tank resistance and reactance (series LC plate tank)

Appendix A: Derivation Of The Previously Used Equations

Let us look at the equivalent circuit given in figure 2. Note that the current arrows arbitrarily define the positive current direction. By virtue of Kirchhoff's current law [9] we get

$$I_0 + I_f = I_C \tag{10}$$
$$I_f + I_2 = I_P \tag{11}$$

When applying Kirchhoff's voltage law, care must be taken to select voltage loops along paths that have a consistent definition of positive current direction[17]. We then obtain

$$U_L + U_R + U_C = U_0 \tag{12}$$
$$U_{RP} + U_{Z2} = -\mu U_C \tag{13}$$
$$U_{RP} + U_{Cf} + U_C = -\mu U_C \tag{14}$$

[17] In other words: The voltage loop must follow the current arrows

combining equations (13) and (14) leads to

$$U_{Z2} - U_{Cf} - U_C = 0$$

which is used to replace equation (14) in the above set of equations. Using Ohm's law for complex impedances then yields

$$j\omega L I_0 + R I_0 + \frac{1}{j\omega C} I_C = U_0 \tag{15}$$

$$R_P I_P + Z_2 I_2 = -\frac{\mu}{j\omega C} I_C \tag{16}$$

$$Z_2 I_2 - \frac{1}{j\omega C_f} I_f - \frac{1}{j\omega C} I_C = 0 \tag{17}$$

using equation (10) to eliminate I_C we get

$$j\omega L I_0 + R I_0 + \frac{1}{j\omega C}(I_0 + I_f) = U_0 \tag{18}$$

$$R_P I_P + Z_2 I_2 = -\frac{\mu}{j\omega C}(I_0 + I_f) \tag{19}$$

$$Z_2 I_2 - \frac{1}{j\omega C_f} I_f - \frac{1}{j\omega C}(I_0 + I_f) = 0 \tag{20}$$

Using the last two equations from the above set along with equation (11) and assuming $\mu \gg 1$ and $C \gg C_f$ leads to

$$I_f = I_0 \frac{-\frac{\mu}{j\omega C} - \frac{R_P}{jZ_2\omega C}}{\frac{\mu}{j\omega C} + R_P + \frac{R_P}{jZ_2\omega C_f} + \frac{1}{j\omega C_f}}$$

which can be plugged into equation (18) yielding

$$I_0 \left(j\omega L + R + \frac{1}{j\omega C} + \underbrace{\frac{-\frac{\mu}{j\omega C} - \frac{R_P}{jZ_2\omega C}}{j\omega C \cdot \left(\frac{\mu}{j\omega C} + R_P + \frac{R_P}{jZ_2\omega C_f} + \frac{1}{j\omega C_f}\right)}}_{Z_f} \right) = U_0$$

from which we can immediately identify Z_f. By substituting $Z_2 = \frac{j}{B}$ and splitting Z_f into it's real and imaginary parts (using once more that $\mu \gg 1$ and $C \gg C_f$) we finally arrive at the equations presented in the main section of this paper.

References

[1] http://www.radiomuseum.org/
forumdata/users/133/PDF/RegenerationLC.pdf

[2] http://en.wikipedia.org/wiki/Regenerative_circuit

[3] E. H. Armstrong, "Wireless Receiving System," US Patent No. 1113149, 1913

[4] http://de.wikipedia.org/wiki/Huth-Kühn-Schaltung

[5] http://www.radiomuseum.org/forum/etwas_fuer_audion_enthusiasten.html

[6] http://www.radiomuseum.org/forum/o_van_bronk_verus_eh_armstrong.html

[7] http://en.wikipedia.org/wiki/Q_factor

[8] http://www.radiomuseum.org/forumdata/upload/
Input impedance of a triode with a complex plate load.pdf

[9] http://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws