Relaxation Oscillations In LC-Oscillators

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Abstract

The occurrence of relaxation oscillations in LC-oscillators is studied from a theoretical and experimental point of view with appropriate attention given to the transition between harmonic oscillations and relaxation oscillations. A generic model circuit for a tuned input, inductive feedback LC-oscillator is introduced and a Liénard type differential equation governing it's behavior is derived. After acquiring a basic understanding of relaxation oscillations by studying Van der Pol's differential equation, a diode clipper is introduced into the LC-oscillator tank and it is shown that the resulting oscillator is qualitatively similar to the Van der Pol oscillator. The resulting differential equation is then solved numerically and it is shown that this oscillator can, as expected, exhibit harmonic oscillations as well as relaxation oscillations. Finally, a practical circuit of an LC-oscillator with LC tank diode clipping that can show harmonic oscillations as well as relaxation oscillations is presented.

Introduction

Relaxation oscillations are a frequently encountered phenomenon in nature. In electrical engineering, relaxation oscillators like the astable multivibrator have numerous applications. Figure 1 shows a typical relaxation oscillation in an electronic system producing an output voltage $U(t)$.

The typical pattern of relaxation oscillations is a relatively slow decay of the output starting from the positive extremum, followed by a sudden jump to the negative extremum, followed again by a relatively slow decay and another sudden jump to the positive extremum again at which the cycle then repeats. Obviously, relaxation oscillations are fundamentally different from sinusoidal harmonic oscillations.

Our interest is therefore naturally directed towards systems that are capable of exhibiting both, (approximate) harmonic oscillations and relaxation oscillations. It may come as a surprise that an LC-oscillator is such a system since this type of oscillator is usually associated with harmonic oscillations. We shall therefore in

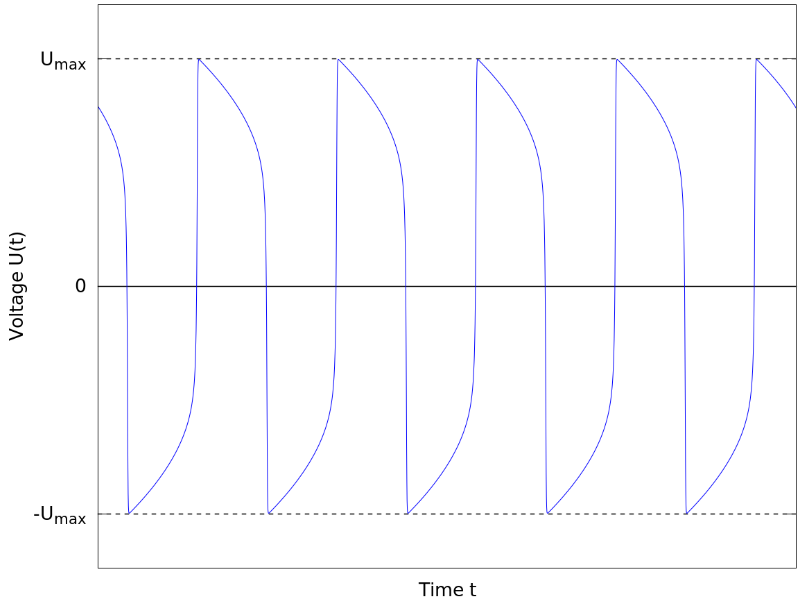

Figure 1: Relaxation oscillations

the remainder of this paper investigate the occurrence of relaxation oscillations in LC-oscillators.

A Generic Tuned Input, Inductive Feedback LC-Oscillator

Let us start with a model of a generic LC-oscillator that has the LC tank at the input side of the feedback device and uses inductive feedback through a feedback coil L_{f} coupled inductively to the tank coil L with a coupling factor of $0 \leq k \leq 1$ as depicted in figure 2.

In this case, it turns out to be convenient to model the losses in the LC tank by a parallel loss resistor R_{P} giving rise to a Q-factor [4] of $Q = R\sqrt{C/L}$. The feedback device "FB" is assumed to be a current source that impresses a current I_{f} into the feedback coil. Also, the feedback device is assumed to have a finite input resistance, giving rise to an input current I_{IN}. The dependency of I_{IN} and I_{f} on the tank voltage U shall be given by

$$\boxed{I_{\mathrm{IN}} = g(U) \qquad \text{and} \qquad I_{\mathrm{f}} = h(U)}$$

where g and h are two sufficiently smooth and, in general, non-linear functions. We shall now derive a differential equation for $U(t)$ governing the circuit. The

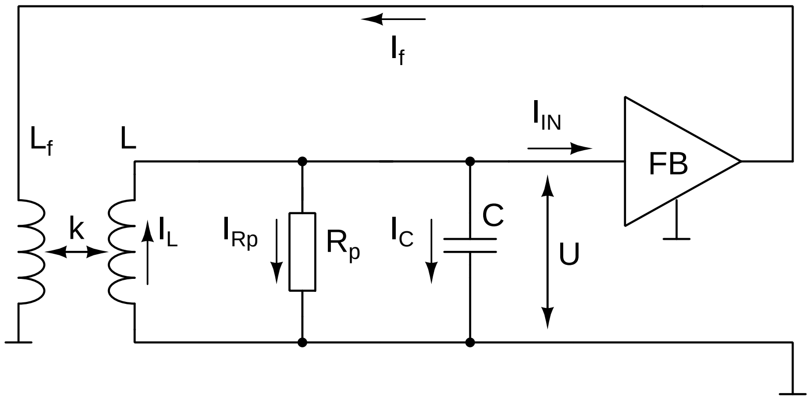

Figure 2: Tuned input, inductive feedback oscillator

reader not interested in the details of the derivation may at this point skip to equation (5).

From Kirchhoff's laws [1] and abiding by the arbitrarily defined positive current directions in figure 2, we obtain

$$I_L = I_{R_\mathrm{P}} + I_C + I_\mathrm{IN} \qquad (1) \qquad\qquad U = U_C = U_{R_\mathrm{P}} = -U_L \qquad (2)$$

Also, for the current I_C through the capacitor C and the voltage U_L across the inductor L we have[1] [2], [3]

$$I_C = C\dot{U}_C \qquad (3) \qquad\qquad U_L = L\dot{I}_L - M\dot{I}_\mathrm{f} \qquad (4)$$

where the mutual inductance M between tank and feedback coil is given by $M = k\sqrt{LL_\mathrm{f}}$. Differentiating equation (1) with respect to time and using equations (3) and (4) then yields

$$\frac{1}{L}U_L + \frac{M}{L}\dot{I}_\mathrm{f} - \frac{1}{R_\mathrm{P}}\dot{U}_{R_\mathrm{P}} - C\ddot{U}_C - \dot{I}_\mathrm{IN} = 0$$

Applying the chain rule of differential calculus to $I_\mathrm{IN} = g(U)$ and $I_\mathrm{f} = h(U)$ gives

$$\dot{I}_\mathrm{IN} = g'(U)\dot{U} \qquad \text{and} \qquad \dot{I}_\mathrm{f} = h'(U)\dot{U}$$

[1]In this paper we shall follow the usual conventions and define $\dot{x} = dx/dt$ and $f'(x) = df(x)/dx$.

which can be plugged into the above differential equation. Using equation (2) we then arrive at the differential equation

$$LC\ddot{U} + L\left(\frac{1}{R_P} + g'(U) - k\sqrt{\frac{L_\mathrm{f}}{L}}h'(U)\right)\dot{U} + U = 0 \tag{5}$$

Introducing the stretched time scale

$$\boxed{t' = \omega_0 t \qquad \text{with} \qquad \omega_0 = \sqrt{\frac{1}{LC}}}$$

which entails $dU/dt = \omega_0 \cdot dU/dt'$ and $d^2U/dt^2 = \omega_0^2 \cdot d^2U/dt'^2$ we finally obtain[2]

$$\ddot{U} + p(U) \cdot \dot{U} + U = 0 \tag{6}$$

with the damping function $p(U)$ given by

$$p(U) = \sqrt{\frac{L}{C}}\left(\frac{1}{R_P} + g'(U) - k\sqrt{\frac{L_\mathrm{f}}{L}}h'(U)\right) \tag{7}$$

The critical damping [5] threshold of equation (6) is readily determined to be $p(U) = 2$.

It is no surprise that equation (6) is a Liénard type differential equation [6] since this type of differential equation frequently arises with the study of oscillatory systems. Liénard type differential equations have been extensively studied regarding existence and uniqueness of stable limit cycles for arbitrary initial conditions $U(0) = \xi_1$, $\dot{U}(0) = \xi_2$.

Understanding Relaxation Oscillations

Relaxation Oscillations were first studied in the 1920's by Van der Pol and many others [7] using the differential equation

$$\ddot{U} + \epsilon\left(U^2 - 1\right) \cdot \dot{U} + U = 0$$

which is nowadays known as the Van der Pol equation. Obviously, the Van der Pol equation is also a Liénard type differential equation with a damping function $p(U)$ of

$$p(U) = \epsilon\left(U^2 - 1\right) \tag{8}$$

[2] Note that from now on, the "dot notation" designates differentiation with respect to t'.

For the case of $\epsilon \ll 1$ the Van der Pol equation produces nearly harmonic oscillations with a period of approximately $T = 2\pi$. However, for the case of $\epsilon \gg 1$ the Van der Pol equation exhibits pronounced relaxation oscillations [8] and we shall therefore take a look at the Van der Pol damping function $p(U)$ for $\epsilon = 10$ as depicted in figure 3 from which we can immediately explain the typical relaxation oscillation curve as shown in figure 1.

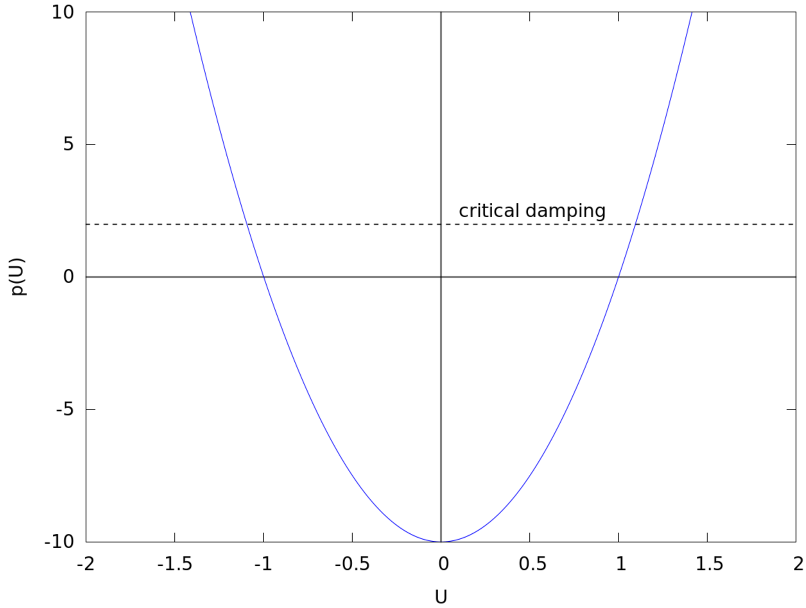

Figure 3: Van der Pol damping for $\epsilon = 10$

Let's assume U is at $-U_{\text{max}}$ in the left ($U < 0$) high damping region of figure 3 and will now start to move back towards $U = 0$. Due to the high over-critical damping in this region, this motion[3] is slow at first. However, after some time, U is leaving the high damping region of $p(U)$ and moves increasingly fast towards $U = 0$. As it enters the strong negative damping region around $U = 0$ it is catapulted into the right ($U > 0$) high damping region of $p(U)$ and stops at U_{max}. It then slowly (due to the high over-critical damping) leaves the right high damping region of $p(U)$ and approaches $U = 0$ again. As it enters the strong negative damping region around $U = 0$ it is catapulted into the left high damping region towards $-U_{\text{max}}$ and the whole cycle starts again.

In contrast, the nearly harmonic oscillations in case of $\epsilon \ll 1$ arise due to the fact that for small values of ϵ the negative damping around $U = 0$ is not strong enough to catapult $U(t)$ into the high damping regions and the occurring weak

[3]Bear in mind that the LC-oscillator is mathematically equivalent to a one-dimensional mechanical oscillator with U being the space coordinate.

damping will only have a minute deformation effect on the otherwise harmonic oscillations. In this context, it is also noteworthy that the relaxation oscillations exhibited by the Van der Pol equation have a period much longer than the period of $T = 2\pi$ in the nearly harmonic case for $\epsilon \ll 1$. This can be attributed to the relatively large amount of time spent in the high damping regions of $p(U)$.

From this understanding of how relaxation oscillations work, we can assume that any system governed by equation (6) that has a damping function $p(U)$ that is qualitatively similar to the Van der Pol damping function depicted in figure 3 can exhibit harmonic oscillations with a period of approximately $T = 2\pi$ (on a stretched time scale of $t' = \omega_0 t$) as well as relaxation oscillations with a much longer period. Those relaxation oscillations will occur if the negative damping around $U = 0$ is strong enough to launch U well into the high damping regions of $p(U)$ in each cycle.

LC Relaxation Oscillations Caused By Diode Clipping

As the reader can easily verify, using a linear feedback device with transconductance $g_m = a$ that has an input characteristic of $I_{\text{IN}} \propto U^3$ will yield a Van der Pol type damping function given by equation (8)[4]. Hence, using such a feedback device would enable the circuit from figure 2 to perform nearly harmonic oscillations as well as relaxation oscillations based on the choice of the transconductance parameter $a = g_m$.

Here, we shall however focus on a more realistic input characteristic $I_{\text{IN}} = g(U)$ of the feedback device and assume that the input of the feedback device itself has a very high impedance but also needs to be protected from voltages above a critical threshold[5] using a two-way diode clipper as depicted in figure 4.

Assuming that we use N identical diodes with an ideality factor of n in each branch, the voltage drop U_{d} across each diode is $U_{\text{d}} = U/N$. Since the forward current of one branch is added to the reverse current of the other, by virtue of Shockley's diode equation [9] we obtain

$$I_{\text{IN}} = g(U) = I_s \left[\exp\left(\frac{U/N}{nU_{\text{th}}} \right) - 1 \right] - I_s \left[\exp\left(\frac{-U/N}{nU_{\text{th}}} \right) - 1 \right]$$

[4]This result is readily obtained by using $h(U) = aU$ along with $I_{\text{IN}} = g(U) = (1/3)bU^3$ in equation (7). A Van der Pol type damping function as given by equation (8) with $\epsilon = b\sqrt{L/C}$ is then obtained by choosing $b = ak\sqrt{L_{\text{f}}/L} - 1/R_{\text{P}}$.

[5]A practical example of such a device is a MOSFET.

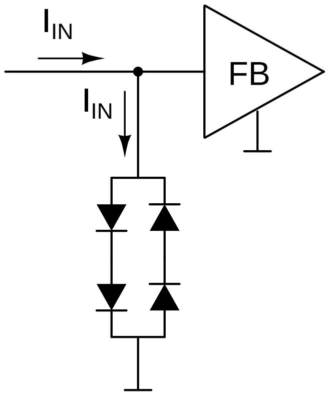

Figure 4: Diode clipper as voltage limiter

where I_s is the reverse bias saturation current of the diode and U_{th} is the thermal voltage[6]. Note that the minus sign between the two terms arises due to the fact that Shockley's equation correctly gives the reverse current with a negative sign for negative voltages. The expression above readily simplifies to

$$g(U) = I_s \left[\exp\left(\frac{U}{nNU_{\mathrm{th}}}\right) - \exp\left(\frac{-U}{nNU_{\mathrm{th}}}\right) \right]$$

from which we straight forwardly obtain

$$\boxed{g'(U) = \frac{I_s}{nNU_{\mathrm{th}}} \left[\exp\left(\frac{U}{nNU_{\mathrm{th}}}\right) + \exp\left(\frac{-U}{nNU_{\mathrm{th}}}\right) \right]} \tag{9}$$

Again, we assume that the feedback device is linear on the output side with a transconductance of $g_m = a$, hence $h(U) = a \cdot U$ from which we immediately obtain

$$\boxed{h'(U) = a} \tag{10}$$

We can now plug these expressions for $g'(U)$ and $h'(U)$ into equation (7) to obtain the damping function $p(U)$ for the LC-oscillator with a diode clipper across the tank. To get an idea of it's shape, $p(U)$ has been plotted in figure 5 for two diodes ($N = 2$) in each branch with a reverse bias saturation current of $I_s = 1\mathrm{pA}$ and an ideality factor of $n = 1.5$. The remaining parameters are as follows: $U_{\mathrm{th}} = 26\mathrm{mV}$, $R_{\mathrm{P}} = 150\mathrm{k}\Omega$, $L = 400\mu\mathrm{H}$, $C = 100\mathrm{pF}$, $L_{\mathrm{f}} = 100\mu\mathrm{H}$, $k = 0.7$ and $a = 15\mathrm{mS}$.

[6] Approximately 26mV at room temperature.

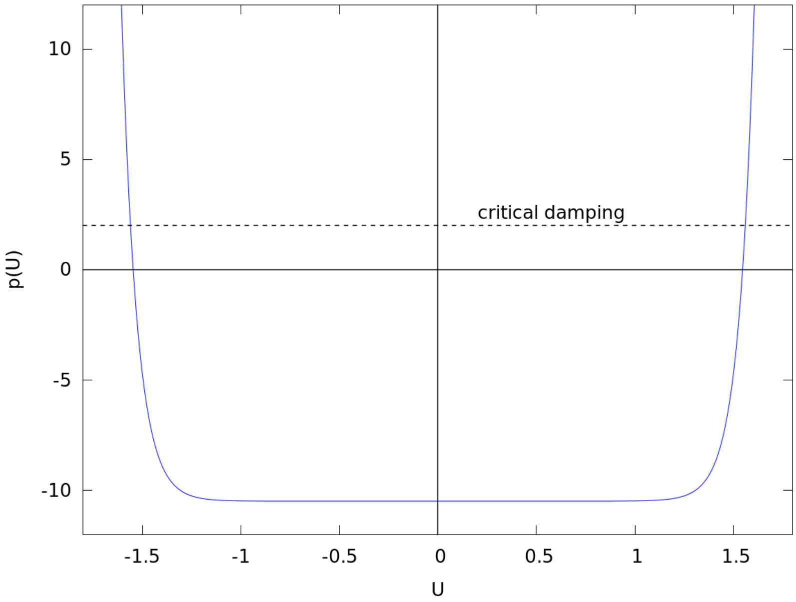

Figure 5: Damping due to diode clipping

Since the resulting damping function is qualitatively similar to the Van der Pol damping function shown in figure 3, we expect the circuit to show nearly harmonic oscillations for low transconductances and relaxation oscillations for higher transconductances of the feedback device.

Indeed, using LSODE [10] to numerically solve differential equation (6) governing the circuit with the parameters given above, we find nearly harmonic oscillations with a period of $T = 2\pi \cdot 1.0017$ on the stretched time scale for a transconductance of $g_m = a = 0.1\text{mS}$ and pronounced relaxation oscillations with a period of $T = 2\pi \cdot 4.096$ on the stretched time scale for $g_m = a = 10\text{mS}$ as can be seen from figure 6.

In practical circuits, diode clipping can often occur implicitly without the presence of a diode clipper across the LC tank. A typical example would be a JFET stage where the gate voltage rises above the source voltage.

A Practical Circuit

After looking at relaxation oscillations from a theoretical point of view, let us now design a practical circuit that can be modeled by our generic circuit from figure 2 and includes a diode clipper as depicted in figure 4 to experimentally explore the transition from nearly harmonic to relaxation oscillations.

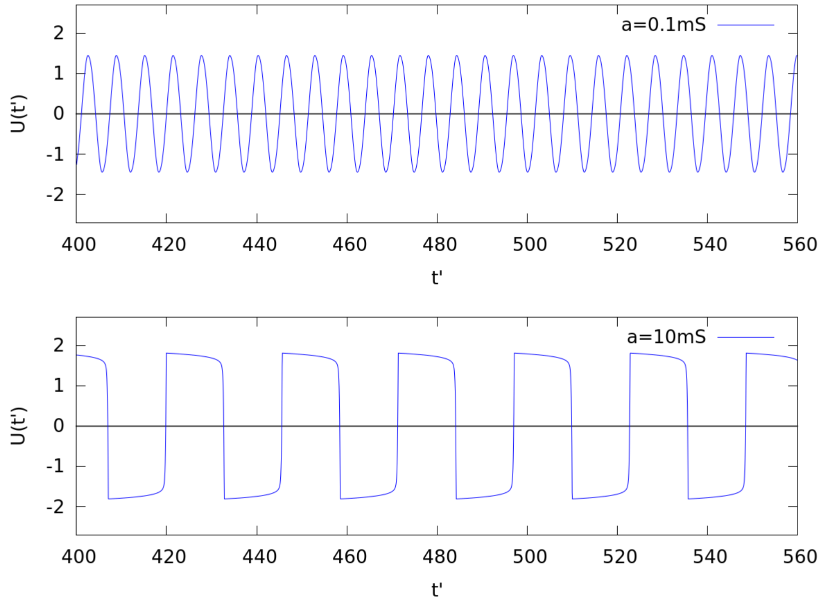

Figure 6: LC-oscillator harmonic and relaxation oscillations

Going over the assumptions on the feedback device from the previous section again, we see that our main task is to design a feedback device that, at least approximately, has the following properties:

- Acts as a linear voltage controlled current source with high output resistance
- Has a reasonably high and constant input resistance
- Provides a simple way of setting the desired transconductance

These requirements can be fulfilled by a BJT emitter follower stage that is not used as a voltage buffer but as a voltage controlled current source. The resulting circuit is shown in figure 7.

In this configuration, the emitter voltage U_{E} of the BJT follows the base voltage U_{B} at a distance that is approximately the base-emitter on voltage[7] U_{on}. Hence, we have $\Delta U_{\mathrm{B}} = \Delta U_{\mathrm{E}} = R_{\mathrm{E}} \Delta I_{\mathrm{E}}$. Since for any reasonably high current gain[8] β of the BJT the collector current I_{C}, which is also the feedback current I_{f}, is approximately equal to the emitter current I_{E}, we get

[7] Typically 0.65V for the BC547C.

[8] The minimum DC current gain of the BC547C is given as $\beta = 420$ in the datasheet. However, one has to take into account that β decreases at higher frequencies. The transit frequency, for which β has dropped to $\beta = 1$, is given as $f_{\mathrm{t}} = 100$MHz in the datasheet of the BC547C.

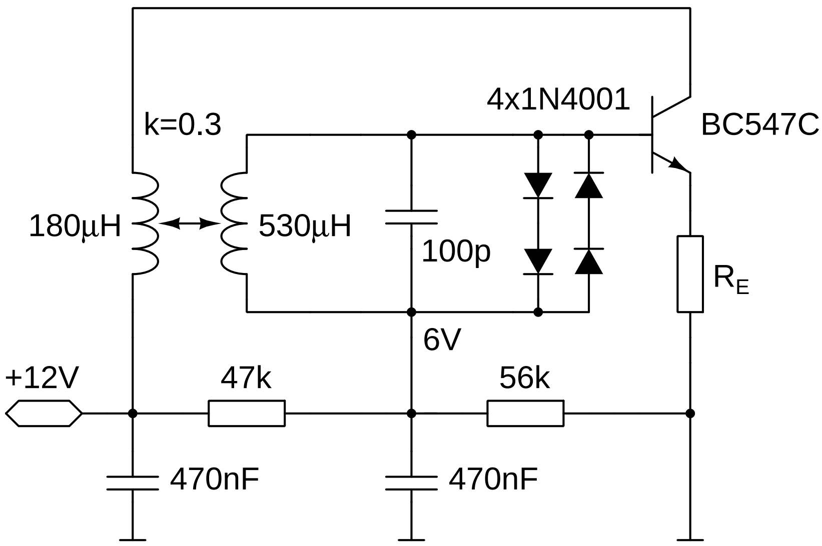

Figure 7: Practical LC-oscillator with amplitude limiting

$$\boxed{g_m = \frac{\Delta I_{\mathrm{f}}}{\Delta U_{\mathrm{B}}} = \frac{\Delta I_{\mathrm{C}}}{\Delta U_{\mathrm{B}}} \approx \frac{\Delta I_{\mathrm{E}}}{\Delta U_{\mathrm{B}}} = \frac{\Delta U_{\mathrm{B}}/R_{\mathrm{E}}}{\Delta U_{\mathrm{B}}} = \frac{1}{R_{\mathrm{E}}}}$$

which is a very convenient way to set the desired transconductance g_m of the feedback circuit. Also, since the input resistance R_{IN} of a BJT emitter follower is $R_{\mathrm{IN}} = \beta R_{\mathrm{E}}$, the input resistance will be reasonably high if we choose $R_{\mathrm{E}} \geq 500\Omega$.

Setting the transconductance of the feedback circuit to $g_m = 0.1\mathrm{mS}$ by choosing $R_{\mathrm{E}} = 10\mathrm{k}\Omega$ we obtain nearly harmonic oscillations at a frequency of approximately 556kHz[9] as shown in figure 8.

Increasing the transconductance of the feedback circuit will gradually push the oscillator into the region of relaxation oscillations. Figure 9 shows pronounced relaxation oscillations at a significantly lower frequency of approximately 302kHz for a transconductance of $g_m = 1.8\mathrm{mS}$ ($R_{\mathrm{E}} = 560\Omega$).

These experimental results nicely corroborate the theoretical results from the previous sections.

[9]Bear in mind that there's a parasitic capacitance of almost 50pF connected in parallel to the LC tank. This parasitic capacitance is composed mainly of the probe's capacitance, the capacitance of the diode clipper and the base-emitter capacitance of the BC547C.

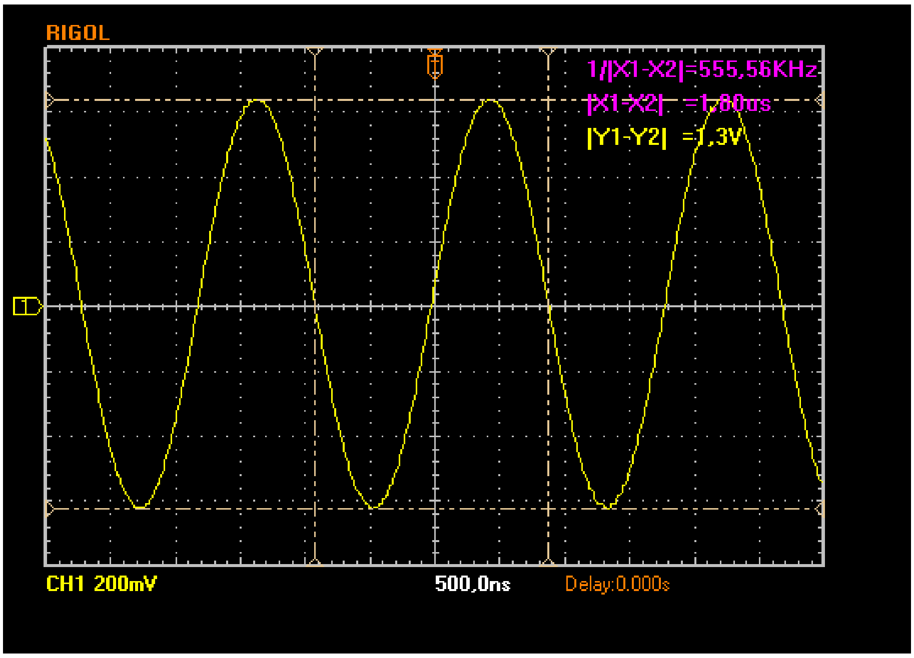

Figure 8: Nearly harmonic oscillations for $R_E = 10\mathrm{k}\Omega$

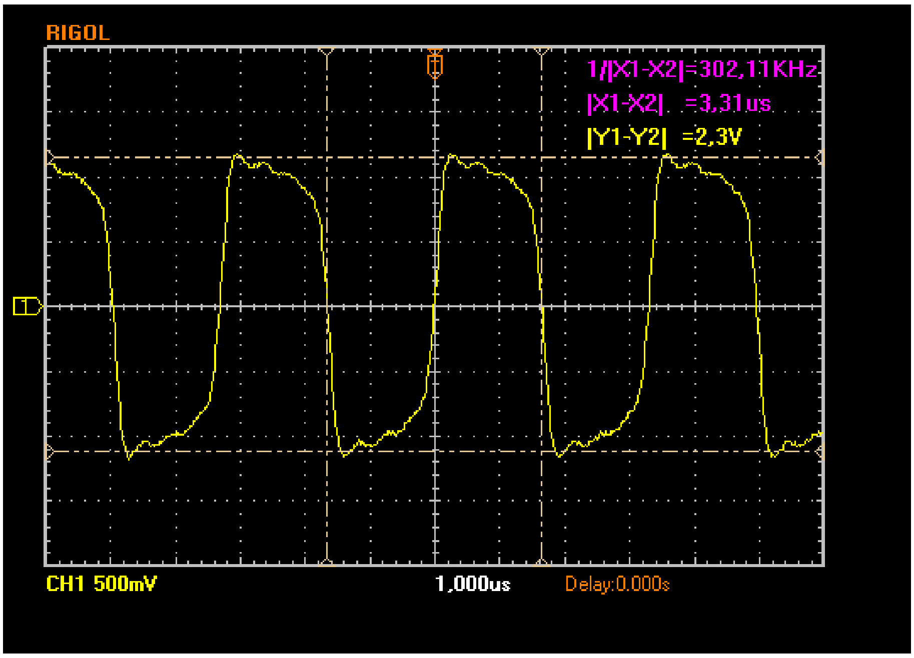

Figure 9: Relaxation oscillations for $R_E = 560\Omega$

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