Exploring Superregeneration II - An Analytic Approach

Dipl.-Phys. Jochen Bauer

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Abstract

This article takes a closer look at the superregenerative receiver invented by Armstrong [3] using an analytic, non-numeric approach. By treating the superregenerative circuit employing a separate quench function generator as a parametric oscillator, well known methods for solving the resulting differential equations can be applied. An approximate solution for the development of oscillation bursts in the circuit for a typical time-dependent feedback is given and analyzed in detail. In particular, expressions for the frequency response curve and bandwidth as well as the amplification factor are given.

Note: The reader should already have read the predecessor of this paper [1] or should be otherwise familiar with the basic principles of superregeneration.

The Superregenerative Circuit As A Parametric Oscillator

In the predecessor of this paper [1] the differential equation for the charge Q of the capacitor C of the tuned circuit in a superregenerative receiver based on a linear feedback model has been shown to be

$$L\ddot{Q}(t) + \tilde{R}(t)\dot{Q}(t) + \frac{1}{C}Q(t) = U_A(t) \tag{1}$$

provided that the quench cycle frequency is considerably lower than the resonant frequency of the tuned circuit. In the above expression, L is the inductance of the tuned circuit, $U_A(t)$ is the driving voltage (antenna input) and $\tilde{R}(t)$ is the time-dependent virtual (series) loss resistance depending on the physical loss resistance of the tuned circuit and the amount of positive feedback applied [1]. In a superregenerative receiver relying on external quenching, the amount of positive feedback and therefore the virtual loss resistance $\tilde{R}(t)$ governing the quench cycle of the superregenerative circuit is controlled by a separate function generator module in the circuit.

[1]For a detailed explanation, the reader is referred to [1] and the references given therein.

Using $Q(t) = CU(t)$ where $U(t)$ is the voltage across the capacitor and $\omega_0^2 = 1/LC$, this differential equation can be brought into the more convenient form

$$\boxed{\ddot{U}(t) + \frac{1}{L}\tilde{R}(t)\dot{U}(t) + \omega_0^2 U(t) = \omega_0^2 U_A(t)} \tag{2}$$

The above differential equation describes a parametric oscillator [2] with a constant resonant frequency ω_0. For any reasonable Q-factor of the tuned circuit ($Q \gg 2$) and provided that

$$\frac{d}{dt}\tilde{R}(t) \ll 2L\omega_0^2 \tag{3}$$

i.e. $\tilde{R}(t)$ is only changing slowly with respect to the resonant frequency of the tuned circuit [2], we can apply well known mathematical procedures (see appendix A) for differential equations of this type. The solution of (2) can then be written as

$$U(t) = U_h(t) + U_p(t) \tag{4}$$

$$U_h(t) = (c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t)) \cdot e^{-\frac{1}{2L}\int_a^t \tilde{R}(\tau)d\tau} \tag{5}$$

$$U_p(t) = \omega_0 \left(\int_b^t U_A(\tau) e^{\frac{1}{2L}\int_a^\tau \tilde{R}(\tau')d\tau'} \cdot \sin(\omega_0(t-\tau))\, d\tau \right) \cdot e^{-\frac{1}{2L}\int_a^t \tilde{R}(\tau)d\tau} \tag{6}$$

In the above equations, $U_h(t)$ is the complete solution of the homogeneous differential equation ($U_A(t) \equiv 0$) while $U_p(t)$ is a particular solution, also called a particular integral, of the inhomogeneous differential equation ($U_A(t)$ non-zero for at least one t). Let us now look at the two parts of the general solution individually.

Obviously, $U_h(t)$ describes the free oscillations of the system with initial conditions given by c_1 and c_2. These free oscillations are responsible for a "memory effect" in the circuit that was demonstrated numerically in [1] were free oscillation bursts ($U_A(t) \equiv 0$) created by non-zero initial conditions in the tuned circuit at $t = a$ when the system is switched on keep coming back each quench cycle provided that the average virtual loss resistance

$$\bar{\tilde{R}} = \frac{1}{T_q}\int_0^{T_q} \tilde{R}(\tau)d\tau$$

over one quench period T_q is zero or negative. In fact, if the average virtual loss resistance is equal to zero, the free oscillation burst will be coming back at

[2] These conditions also need to be fulfilled to arrive at equation (1) in the first place, see [1] for details.

an unchanged maximum amplitude, while for an average virtual loss resistance below zero, the maximum amplitude will even increase exponentially over time. Obviously, for the regenerative receiver to function properly, $\tilde{R}(t)$ needs to be chosen so that it's average is sufficiently above zero, resulting in an exponential decay of the free oscillations given by $U_h(t)$. The reader is again referred to appendix A for a detailed proof.

At this point, some clarification on superregeneration and "free oscillations" seems appropriate. In our approach here, the "free oscillations" responsible for an undesired memory effect in the circuit are the non-driven oscillatory solutions of the differential equations governing the *parametric oscillator* with a continuously variable virtual loss resistance and are not to be confused with the "free oscillations" invoked in simplified explanations of superregeneration relying on a simple *non-parametric* tuned circuit respectively oscillator as given by Armstrong [3] and repeated in the predecessor of this paper [1].

Let us now turn to the solutions $U_p(t)$, of the driven system. Since, as we have seen above, $U_h(t)$ needs to decay exponentially, the only remaining part of the solution is the particular integral $U_p(t)$. First, note that $U_p(t)$ is independent of the parameter a as shown in appendix A. However, strictly speaking, there is an infinite number of particular integrals $U_p(b, t)$, each of them given by a specific value of b and the reader might wonder which one to pick. Fortunately, for any reasonable function $\tilde{R}(t)$ that has a linear approximation around $\tilde{R} = 0$ when crossing into the negative region, it turns out that all particular integrals $U_p(b, t)$ are approximately equal and independent of b. The reader is once again referred to appendix A for details.

A Simple Time-Dependent Virtual Loss Resistance

In order to evaluate $U(t) \approx U_p(t)$ from equation (6) we need to specify the virtual loss resistance $\tilde{R}(t)$ as a function of time. Since we are mostly interested in the build-up of the oscillations in the tuned circuit and not so much in their successive quenching, we only need an expression for $\tilde{R}(t)$ for the first part of the quenching cycle. Hence, let us assume that $\tilde{R}(t)$ descends linearly around $t = 0$ from the positive region into the negative region. More precisely, it should start the linear phase of it's descent at $t = -t_1$, cross $\tilde{R} = 0$ at $t = 0$ and end it's linear descent at $t = t_1$, now taking a constant value of $\tilde{R}(t) = -R_1$ for $t_1 \leq t \leq t_2$. For $t > t_2$, we require $\tilde{R}(t)$ to quickly move up well into the positive region and to stay there to quench the oscillations before the next cycle begins. If the oscillations from the previous quench cycle die down to a neglectable amplitude before the beginning of the next quench cycle, each cycle may be treated independently from it's predecessor and the following results are applicable to all quench cycles. Let us now look at a sketch of $\tilde{R}(t)$ as described above (figure 1).

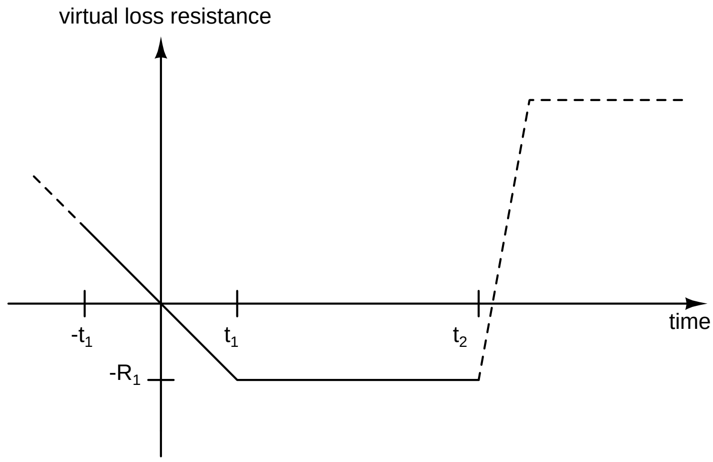

Figure 1: Virtual loss resistance $\tilde{R}$ as a function of time

In the range of $-t_1 \le t \le t_2$, this virtual loss resistance function $\tilde{R}(t)$ can be written as

$$\tilde{R}(t) = \begin{cases} -mt & ; -t_1 \le t \le t_1 \\ -R_1 & ; t_1 < t \le t_2 \end{cases} \tag{7}$$

It immediately becomes clear that the negative slope m of $\tilde{R}(t)$ around $\tilde{R} = 0$ is given by $m = R_1/t_1$.

The Closed-Form Solution

Using $\tilde{R}(t)$ as defined in equation (7) in equation (6) and setting $a = b = -t_1$ as well as assuming the driving voltage $U_A(t)$ to be a sinusoid with amplitude $\hat{U}_A$, angular frequency ω and arbitrary phase angle φ, i.e.

$$U_A(t) = \hat{U}_A \sin(\omega t + \varphi) \tag{8}$$

the following approximation of $U(t)$ valid for $t_1 + 8L/R_1 \le t \le t_2$ and $m \ge 16L/t_1^2$ can be obtained:

$$\boxed{U(t) \approx -\hat{U}_A e^{\frac{R_1}{2L}(t-t_1)} e^{\frac{m}{4L}t_1^2} \omega_0 \sqrt{\frac{\pi L}{m}} e^{-\frac{L}{m}(\omega-\omega_0)^2} \cos(\omega_0 t + \varphi)} \tag{9}$$

It should be noted that the above formula works very well for times t where $|U(t)|$ is relatively large but is less accurate for times t where $|U(t) \approx 0|$. This is however of no concern to us since we are interested in the amplitude of the oscillations of $U(t)$ and not how it behaves around $U \approx 0$. As usual, a detailed derivation of the above results can be found in appendix A.

At this point, let us consider the following example to get an idea of the magnitude of all the quantities involved in the equations so far. We shall consider a superregenerative receiver designed for a frequency of around 30MHz and therefore employing a coil of $L = 3\mu$H. First, we can use equation (3) to obtain the maximum negative slope $m_{\max}$ that our solution is valid for. We immediately obtain

$$\frac{d}{dt}\tilde{R}(t) = m \ll 213 \frac{\mathrm{G}\Omega}{\mathrm{s}}$$

Hence, we can set $m_{\max} = 20\mathrm{G}\Omega/\mathrm{s}$. Next, we need to get an idea of the magnitude of t_1. Suppose that the receiver is designed for VHF-AM reception. The minimum quench cycle frequency then needs to be twice the desired audio signal bandwidth of the receiver [11]. If we settle for an audio bandwidth of 4kHz, the maximum quench cycle period is obviously $T_q = 125\mu$s. Of this quench cycle time, we need to save a considerable portion for the part of the cycle where $\tilde{R} = -R_1$ and, of course, the part of the cycle where the oscillations are quenched. A conservative setting for t_1 is therefore $t_1 = 20\mu$s.

We can now, by virtue of $m \geq 16L/t_1^2$ obtain the minimum negative slope $m_{\min}$ for which our solution is valid. The result is $m_{\min} = 120\mathrm{k}\Omega/\mathrm{s}$. With these results, we can then calculate $R_1 = m \cdot t_1 = 2.4\Omega$ and the minimum time t for equation (9) to be applicable is then $t \geq t_1 + 8L/R_1 = 30\mu$s.

Bandwidth, Gain And Linearity

Let us move from right to left in interpreting $U(t)$ as given by equation (9). Obviously, the factor $\cos(\omega_0 t + \varphi)$ is the oscillatory part while the other factors give the envelope of the oscillations. As far as the oscillatory part goes, it is interesting to note that the arbitrary phase angle φ of the input signal also appears in the oscillation bursts of the superregenerative circuit. Next comes the frequency response function

$$g(\omega) = e^{-\frac{L}{m}(\omega-\omega_0)^2} \tag{10}$$

which, unlike the frequency response curve of a non-superregenerative single tank receiver [3] has the shape of a Gaussian Function [5]. From the above expression, the -3dB Bandwidth of the superregenerative receiver is readily obtained by solving [4]

[3] See [4] for generalized frequency response curves of single tank circuits.
[4] Note that $g(\omega)$ is already conveniently normalized on the vertical axis.

$$g(\omega) = \frac{1}{\sqrt{2}}$$

for ω, yielding the solutions

$$\omega_{1/2} = \omega_0 \mp \sqrt{\frac{m}{L}\frac{1}{2}\ln 2}$$

From which the absolute bandwidth $\Delta\omega$ follows to be

$$\boxed{\Delta\omega = \omega_2 - \omega_1 = \sqrt{2\ln 2} \cdot \sqrt{\frac{m}{L}} \approx 1.18 \cdot \sqrt{\frac{m}{L}}} \tag{11}$$

Obviously, neither frequency response nor bandwidth of the superregenerative receiver depend on the Q-factor of the tuned circuit any more [5]. This is in stark contrast to the behavior of non-superregenerative single tank receivers.

It quickly becomes clear that the bandwidth of the superregenerative receiver is relatively large. Even when setting $m = m_{\min} = 120\text{k}\Omega/\text{s}$ and $L = 3\mu\text{H}$ from the previous section, we obtain a bandwidth of $\Delta\omega = 236\text{kHz}$.

Let us now turn to the amplification factors in equation (9). We notice that there is one non-exponential amplification factor

$$\mu_r = \omega_0 \sqrt{\frac{\pi L}{m}}$$

that we might call the regular or non-superregenerative gain factor. It increases as the negative slope of the descent of the virtual loss resistance $\tilde{R}(t)$ into the negative region decreases and the tuned circuit therefore spends more time in the $\tilde{R} \approx 0$ region. Also, by looking at equation (11), we see that it exhibits a reciprocal relation with the bandwidth of the superregenerative receiver, i.e. a higher regular gain factor will entail a smaller bandwidth which is a well known behavior of (regular) regenerative circuits.

Using $m = 120\text{k}\Omega/\text{s}$ and $L = 3\mu\text{H}$ along with $\omega_0 = 2\pi \cdot 30\text{MHz}$ as in the examples before, we obtain $\mu_r \approx 1670$.

Moving further left in equation (9) we now arrive at the exponential gain factor

$$\mu_s = e^{\frac{R_1}{2L}(t-t_1)} e^{\frac{m}{4L}t_1^2}$$

Obviously, this gain factor is due to the exponential build-up of oscillations in the tuned circuit and can therefore be dubbed the superregenerative gain factor. The right exponential function in the above expression can be attributed to the

[5] This has also been demonstrated numerically in the predecessor of this paper [1].

superregenerative gain during the descent of the virtual loss resistance $\tilde{R}(t)$ into the negative region, while the left exponential function accounts for the gain while $\tilde{R}(t) = -R_1$.

We can continue our numeric example used so far and set $m = 120\text{k}\Omega/\text{s}$, $L = 3\mu\text{H}$, $t_1 = 20\mu\text{s}$ and $R_1 = 2.4\Omega$. However, we'll also need a time t at which we look at the amplitude of the oscillations. Setting $t = 60\mu\text{s}$, which is a little before the middle of the quench cycle, just before the oscillation quenching sets in, we obtain $\mu_s \approx 485 \cdot 10^6$. This is clearly an amplification factor that would be impossible to reach by a non-superregenerative single tank circuit.

Finally, as going from right to left in equation (9), we come to the amplitude $\hat{U}_A$ of the driving voltage $U_A(t)$ that is the input signal of the superregenerative circuit. It is obvious that the amplitude of the oscillation bursts is proportional to the amplitude of the input signal. Also, by substituting $U_A(t) = \hat{U}_A \sin(\omega t + \varphi)$ into equation (6) it becomes clear that this holds true for any virtual loss resistance function $\tilde{R}(t)$ that satisfies the necessary prerequisites for the approximations made in this article. However, it needs to be pointed out that in practical circuits that employ non-ideal feedback devices this result only holds true if the amplitude of the oscillations is at all times small enough for the non-ideal feedback device to stay in it's linear region [6].

Appendix A: Deriving The Equations

We start with equation (2). Using the substitution [2]

$$U(t) = y(t) \cdot e^{-\frac{1}{2L}\int_a^t \tilde{R}(\tau)d\tau} \tag{12}$$

yields the following differential equation for $y(t)$

$$\ddot{y}(t) + \left(\omega_0^2 - \frac{1}{2L}\dot{\tilde{R}}(t) - \frac{1}{4L^2}\tilde{R}^2(t)\right) y(t) = \omega_0^2 U_A(t) \cdot e^{\frac{1}{2L}\int_a^t \tilde{R}(\tau)d\tau} \tag{13}$$

Let us first look at the expression $\tilde{R}^2/4L^2$. Making the reasonable assumption that the absolute value $|\tilde{R}|$ of any occurring negative virtual loss resistance does not exceed 4 times the (positive) physical loss resistance R of the tuned circuit and using $\omega_0^2 = 1/LC$ as well as [6]

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

were Q is the Q-factor of the tuned circuit, we have

[6] Typically, some form of automatic gain control will have to be used to limit the amplitude of the oscillation bursts thereby making the feedback device non-linear for larger amplitudes.

$$\frac{1}{4L^2}\tilde{R}^2 \leq \frac{4}{Q^2}\omega_0^2$$

Therefore, for any reasonable Q-factor of $Q \gg 2$, the expression $\tilde{R}^2/4L^2$ can be neglected against ω_0^2 in equation (13).

Let us now look at $\frac{1}{2L}\dot{\tilde{R}}$ from equation (13). We can neglect this expression against ω_0^2 if

$$\dot{\tilde{R}} \ll 2L\omega_0^2$$

thus arriving at

$$\ddot{y}(t) + \omega_0^2 y(t) = f(t) \tag{14}$$

with

$$f(t) = \omega_0^2 U_A(t) \cdot e^{\frac{1}{2L}\int_a^t \tilde{R}(\tau)d\tau}$$

The general solution of the homogeneous version of differential equation (14) ($f(t) \equiv 0$) is then given by

$$y_h(t) = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t)$$

while by virtue of Leibniz' Integral Rule [7]

$$y_p(t) = \frac{1}{\omega_0}\int_b^t f(\tau)\sin\left(\omega_0(t-\tau)\right)d\tau$$

can be shown to be a particular integral of differential equation (14). Substituting the results obtained so far into equation (12) then yields the results given in the main section of this paper.

Let's now take a closer look at the behavior of $U_h(t)$ from equation (5). Since $\tilde{R}(t)$ is T_q-periodic, where T_q is the duration of one quench cycle, it's average over one period is

$$\bar{\tilde{R}} = \frac{1}{T_q}\int_0^{T_q} \tilde{R}(\tau)d\tau$$

We can then write $\tilde{R}(t)$ as

$$\tilde{R}(t) = \tilde{R}_0(t) + \bar{\tilde{R}} \tag{15}$$

Obviously, $\tilde{R}_0(t)$ is also T_q-periodic with it's integral over one period being zero. Let us define the integral function $h(t)$ to be

$$h(t) = \int_a^t \tilde{R}_0(\tau) d\tau$$

for $t \geq a$. Since

$$\begin{aligned} h(t+T_q) &= \int_a^{t+T_q} \tilde{R}_0(\tau) d\tau = \int_a^t \tilde{R}_0(\tau) d\tau + \int_t^{t+T_q} \tilde{R}_0(\tau) d\tau \\ &= \int_a^t \tilde{R}_0(\tau) d\tau + \int_0^{T_q} \tilde{R}_0(\tau) d\tau = \int_a^t \tilde{R}_0(\tau) d\tau + 0 = h(t) \end{aligned}$$

the integral function $h(t)$ is obviously also T_q-periodic. Substituting equation (15) into equation (5) gives

$$U_h(t) = (c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t)) \cdot e^{-\frac{1}{2L} h(t)} \cdot e^{-\frac{1}{2L} \bar{\tilde{R}}(t-a)}$$

from which the claims about the memory effect in the circuit due to $U_h(t)$ made in the main section of this paper follow immediately.

Let us now turn to $U_p(t)$ as given by equation (6). Let $S(t)$ be an anti-derivative of $\tilde{R}(t)$, equation (6) can then be written as

$$\begin{aligned} U_p(t) &= \omega_0 \left(\int_b^t U_A(\tau) e^{\frac{1}{2L} \int_a^\tau \tilde{R}(\tau') d\tau'} \cdot \sin(\omega_0(t-\tau)) d\tau \right) \cdot e^{-\frac{1}{2L} \int_a^t \tilde{R}(\tau) d\tau} \\ &= \omega_0 \left(\int_b^t U_A(\tau) e^{\frac{1}{2L}(S(\tau)-S(a))} \cdot \sin(\omega_0(t-\tau)) d\tau \right) \cdot e^{-\frac{1}{2L}(S(t)-S(a))} \\ &= \omega_0 e^{-\frac{1}{2L} S(a)} e^{\frac{1}{2L} S(a)} \left(\int_b^t U_A(\tau) e^{\frac{1}{2L} S(\tau)} \cdot \sin(\omega_0(t-\tau)) d\tau \right) \cdot e^{-\frac{1}{2L} S(t)} \\ &= \omega_0 \left(\int_b^t U_A(\tau) e^{\frac{1}{2L} S(\tau)} \cdot \sin(\omega_0(t-\tau)) d\tau \right) \cdot e^{-\frac{1}{2L} S(t)} \end{aligned}$$

Obviously, $U_p(t)$ is independent of a.

We shall now evaluate $U_p(t)$ as given in equation (6) with $\tilde{R}(t)$ specified by equation (7) and $U_A(t)$ given by equation (8). Setting $a = b = -t_1$, equation (6) yields

$$U_p(t) = \omega_0 \left(\int_{-t_1}^t \hat{U}_A \sin(\omega\tau + \varphi) e^{\frac{1}{2L} \int_{-t_1}^\tau \tilde{R}(\tau') d\tau'} \cdot \sin(\omega_0(t-\tau)) d\tau \right) \cdot e^{-\frac{1}{2L} \int_{-t_1}^t \tilde{R}(\tau) d\tau}$$

Let us define

$$I_3(t) = e^{-\frac{1}{2L} \int_{-t_1}^t \tilde{R}(\tau) d\tau}$$

and postpone it's evaluation until later. Splitting the remaining integrals into ranges suiting the definition of $\tilde{R}(t)$ from equation (7) and inserting the appropriate expression for $\tilde{R}(t)$ into each integral, we arrive at

$$\begin{aligned} U_p(t) =& \omega_0 \hat{U}_A I_3(t) \int_{-t_1}^{t_1} e^{\frac{1}{2L}\left(\int_{-t_1}^{t_1}(-m\tau')d\tau' + \int_{t_1}^{\tau}(-m\tau')d\tau'\right)} \cdot \sin(\omega\tau+\varphi)\sin\left(\omega_0(t-\tau)\right)d\tau \\ &+ \omega_0 \hat{U}_A I_3(t) \int_{t_1}^{t} e^{\frac{1}{2L}\left(\int_{-t_1}^{t_1}(-m\tau')d\tau' + \int_{t_1}^{\tau}(-R_1)d\tau'\right)} \cdot \sin(\omega\tau+\varphi)\sin\left(\omega_0(t-\tau)\right)d\tau \end{aligned}$$

Performing the inner integrations, we get

$$U_p(t) = \omega_0 \hat{U}_A I_3(t) e^{\frac{m}{4L}t_1^2} \cdot I_1(t) + \omega_0 \hat{U}_A I_3(t) e^{\frac{R_1}{2L}t_1} \cdot I_2(t)$$

with

$$I_1(t) = \int_{-t_1}^{t_1} e^{-\frac{m}{4L}\tau^2} \cdot W(t,\tau)d\tau$$

and

$$I_2(t) = \int_{t_1}^{t} e^{-\frac{R_1}{2L}\tau} \cdot W(t,\tau)d\tau$$

where $W(t,\tau)$ is defined as

$$W(t,\tau) = \sin(\omega\tau+\varphi)\sin\left(\omega_0(t-\tau)\right)$$

Let's now take a closer look at $I_1(t)$. There is no closed-form anti-derivative of the integrand and we'll therefore try to find a closed form approximation of this integral. Since $|W(t,\tau)| \le 1$ and the exponential function

$$e^{-\frac{m}{4L}\tau^2}$$

causes a sharp cut-off of the integrand that is well within the integration boundaries $[-t_1, t_1]$ if

$$\frac{m}{4L}t_1^2 \ge 4 \qquad (16)$$

we may, under the above prerequisite, extend the boundaries of integration to $[-\infty, \infty]$ ending up with

$$I_1(t) \approx \int_{-\infty}^{\infty} e^{-\frac{m}{4L}\tau^2} W(t,\tau)d\tau \qquad (17)$$

This approximation works very well unless $|W(t,\tau)|$ is almost zero within $[-t_1, t_1]$ and grows significantly outside this range, in which case the absolute value of the integral will however be quite small. We therefore note that the above approximation works very well for large values of $|I_1(t)|$ and is less accurate for small values of $|I_1(t)|$. Obviously, in this approximation, $U_p(t)$ is independent of the integral boundaries given in equation (6). It should be noted that this approximation can be applied to all virtual loss resistance functions $\tilde{R}(t)$ that have a linear approximation around $\tilde{R} = 0$.

The validity of extending the integration boundaries may more rigorously be shown by looking at the integral

$$I_1' = \int_{-t_1}^{t_1} e^{-\frac{m}{4L}\tau^2} \cdot 1 \, d\tau$$

Substituting

$$x = \sqrt{\frac{m}{4L}}\tau$$

yields

$$I_1' = \sqrt{\frac{4L}{m}} \int_{-x_1}^{x_1} e^{-x^2} dx = 2\sqrt{\frac{4L}{m}} \int_{0}^{x_1} e^{-x^2} dx$$

with

$$x_1 = \sqrt{\frac{m}{4L}} t_1$$

Since the prerequisite made in equation (16) now translates into $x_1 \geq 2$ we can evaluate I_1' with $x_1 = 2$ and compare the result to I_1' with $x_1 = \infty$. Using numeric tables of Gaussian Integrals we find that

$$I'_{1,x_1=2} = 2\sqrt{\frac{4L}{m}} \cdot \frac{\sqrt{\pi}}{2} \cdot 0.995322265$$

and, of course

$$I'_{1,x_1=\infty} = 2\sqrt{\frac{4L}{m}} \cdot \frac{\sqrt{\pi}}{2} \cdot 1$$

Let us now evaluate $I_1(t)$ as given by equation (17). The function $W(t,\tau)$ can be split into [8]

$$W(t,\tau) = W_1(t,\tau) + W_2(t,\tau) + W_3(t,\tau) + W_4(t,\tau) \tag{18}$$

with

$$
\begin{aligned}
W_1(t,\tau) &= \frac{1}{2}\cos\left((\omega+\omega_0)\tau\right)\cos(\varphi-\omega_0 t)\\
W_2(t,\tau) &= -\frac{1}{2}\sin\left((\omega+\omega_0)\tau\right)\sin(\varphi-\omega_0 t)\\
W_3(t,\tau) &= -\frac{1}{2}\cos\left((\omega-\omega_0)\tau\right)\cos(\varphi+\omega_0 t)\\
W_4(t,\tau) &= \frac{1}{2}\sin\left((\omega-\omega_0)\tau\right)\sin(\varphi+\omega_0 t)
\end{aligned}
$$

Because of the symmetry of $W_2(t,\tau)$ and $W_4(t,\tau)$ with respect to $\tau = 0$ we have

$$
\int_{-\infty}^{\infty} e^{-\frac{m}{4L}\tau^2} W_2(t,\tau)d\tau = \int_{-\infty}^{\infty} e^{-\frac{m}{4L}\tau^2} W_4(t,\tau)d\tau = 0
$$

On the other hand, the integrals involving $W_1(t,\tau)$ and $W_3(t,\tau)$ are evaluated to be [9]

$$
\int_{-\infty}^{\infty} e^{-\frac{m}{4L}\tau^2} W_1(t,\tau)d\tau = \cos(\omega_0 t-\varphi)\cdot\sqrt{\frac{\pi L}{m}}\cdot e^{-\frac{L}{m}(\omega+\omega_0)^2}
$$

and

$$
\int_{-\infty}^{\infty} e^{-\frac{m}{4L}\tau^2} W_3(t,\tau)d\tau = -\cos(\omega_0 t+\varphi)\cdot\sqrt{\frac{\pi L}{m}}\cdot e^{-\frac{L}{m}(\omega-\omega_0)^2}
$$

Since ω and ω_0 have the same magnitude, the integral involving $W_1(t,\tau)$ can obviously be neglected against the integral involving $W_3(t,\tau)$, leaving us with

$$
I_1(t) \approx -\cos(\omega_0 t+\varphi)\cdot\sqrt{\frac{\pi L}{m}}\cdot e^{-\frac{L}{m}(\omega-\omega_0)^2}
$$

The evaluation of $I_2(t)$ is somewhat easier since there is a closed-form anti-derivative of the integrand [10]. Splitting $W(t,\tau)$ again according to equation (18) and performing the integration yields

$$
I_2(t) = J(t) - J(t_1)
$$

with

$$
\begin{aligned}
J(\tau) =& \frac{1}{2}\cos(\varphi-\omega_0 t)\frac{e^{-\frac{R_1}{2L}\tau}}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega+\omega_0)^2}}\cos\left((\omega+\omega_0)\tau-\vartheta_1\right)\\
&-\frac{1}{2}\sin(\varphi-\omega_0 t)\frac{e^{-\frac{R_1}{2L}\tau}}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega+\omega_0)^2}}\sin\left((\omega+\omega_0)\tau-\vartheta_1\right)\\
&-\frac{1}{2}\cos(\varphi+\omega_0 t)\frac{e^{-\frac{R_1}{2L}\tau}}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega-\omega_0)^2}}\cos\left((\omega-\omega_0)\tau-\vartheta_2\right)\\
&+\frac{1}{2}\sin(\varphi+\omega_0 t)\frac{e^{-\frac{R_1}{2L}\tau}}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega-\omega_0)^2}}\sin\left((\omega-\omega_0)\tau-\vartheta_2\right)
\end{aligned}
$$

where

$$
\vartheta_{1/2} = \arccos\left(\frac{-\frac{R_1}{2L}}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega\pm\omega_0)^2}}\right)
$$

If we are only interested in the behavior of the superregenerative circuit at times $t \geq t_1 + 8L/R_1$, we have $J(t) \ll J(t_1)$ and hence $I_2 \approx -J(t_1)$ no longer depends on t. Also, since ω and ω_0 have the same magnitude, all summands in $J(t_1)$ containing $\omega+\omega_0$ can be neglected against their counterparts containing $\omega-\omega_0$ for all reasonable values of R_1 and L. The resulting expression for I_2 then simplifies to

$$
I_2 \approx \frac{1}{2}\frac{e^{-\frac{R_1}{2L}t_1}}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega-\omega_0)^2}}\cos(\varphi-\vartheta_2+\omega t_1)
$$

and we arrive at

$$
\begin{aligned}
U_p(t) \approx& -\omega_0\hat{U}_A I_3(t)e^{\frac{m}{4L}t_1^2}\sqrt{\frac{\pi L}{m}}e^{-\frac{L}{m}(\omega-\omega_0)^2}\cos(\omega_0 t+\varphi)\\
&+\omega_0\hat{U}_A I_3(t)\frac{1}{2}\frac{1}{\sqrt{\left(\frac{R_1}{2L}\right)^2+(\omega-\omega_0)^2}}\cos(\varphi-\vartheta_2+\omega t_1)
\end{aligned}
$$

For all reasonable parameters and $t_1^2 \geq 16L/m$ as required earlier, the second summand of the above expression can safely be neglected against the first summand and we obtain

$$
U_p(t) \approx -\omega_0\hat{U}_A I_3(t)e^{\frac{m}{4L}t_1^2}\sqrt{\frac{\pi L}{m}}e^{-\frac{L}{m}(\omega-\omega_0)^2}\cos(\omega_0 t+\varphi)
$$

Evaluating $I_3(t)$ is easily performed by again splitting the integral into ranges suiting the definition of $\tilde{R}(t)$ from equation (7) and inserting the appropriate expression for $\tilde{R}(t)$ into each integral. The result is

$$I_3(t) = e^{\frac{R_1}{2L}(t-t_1)}$$

and hence, we finally arrive at equation (9).

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